For example, suppose $X(\theta) = \sigma Z + \theta$, where Z is a standard normal random variable. Then $X(\theta)$ is normal with mean θ and standard deviation σ, and $X(\theta)$ is strongly stochastically convex in θ.

A somewhat weaker version of stochastic convexity is the following:

DEFINITION B.4 *$X(\theta)$ is* **stochastically convex in the sample-path sense** *(SCX-sp) if for any four values $\theta_i, i = 1,2,3,4$ satisfying $\theta_2 - \theta_1 = \theta_4 - \theta_3$ and $\theta_4 \geq \max\{\theta_2, \theta_3\}$, there exist random variable $X_i, i = 1,2,3,4$ defined on a common probability space $(\Omega, \mathcal{F}, P)$, such that X_i is equal in distribution to $X(\theta_i)$, $i = 1,2,3,4$ and*

$$X_4(\omega) - X_3(\omega) \geq X_2(\omega) - X_1(\omega),$$

for all $\omega \in \Omega$.

To illustrate, we show that the sum of Bernoulli random variables is stochastically convex (and concave) in this sample path sense. To do so, let $\theta_i, i = 1,2,3,4$ be integers satisfying $\theta_2 - \theta_1 = \theta_4 - \theta_3$ and $\theta_4 \geq \max\{\theta_2, \theta_3\}$, and let ω define an infinite sequence $\{Y_1, Y_2, \ldots\}$ of i.i.d. Bernoulli random variables as before. Note that $\theta_1 \leq \min\{\theta_2, \theta_3\}$ (else $\theta_4 < \max\{\theta_2, \theta_3\}$), and define

$$\begin{aligned}
X_1 &= \sum_{i=1}^{\theta_1} Y_i \\
X_3 &= \sum_{i=1}^{\theta_3} Y_i \\
X_4 &= \sum_{i=1}^{\theta_4} Y_i \\
X_2 &= X_1 + (X_4 - X_3).
\end{aligned}$$

Note X_i is equal in distribution to $X(\theta_i)$ since each is the sum of θ_i i.i.d. Bernoulli random variables, and by construction

$$X_4 - X_3 = X_2 - X_1,$$

so $X(\theta)$ is stochastically convex in the sample path sense.

The following proposition relates these versions of stochastic convexity:

PROPOSITION B.3 *$SSCX \Rightarrow SICX\text{-}sp \Rightarrow SCX$.*

So showing $X(\theta)$ is either strongly stochastically convex or stochastically convex in the sample path sense, implies that $X(\theta)$ is stochastically convex. Again, returning to our example, this implies that if $X(\theta)$ is the sum of θ i.i.d. Bernoulli random variables and $g(x)$ is a convex function, the $E[g(X(\theta))]$ is convex in θ.